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ON THE STABILITY OF POINCARÉ PERIODIC SOLUTIONS OF HARMILTONIAN SYSTEMS*

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Using methods of the theory of stability of equilibrium positions of Hamiltonian systems /1—3/, sufficient conditions are obtained for the orbital stability of the Poincaré periodic solutions of autonomous Hamiltonian systems with two degrees of freedom on the assumption that the unperturbed system is non-degenerate.

1. Consider an autonomous system with two degrees of freedom whose Hamilton function has the form

$$F = F_0(I) + \mu F_1(I, \varphi)^+ \ldots, \varphi \in \mathbf{T}^2, I \in Q \tag{1.1}$$

where $\varphi = (\varphi_1, \varphi_2)$ are generalized coordinates, $I = (I_1, I_2)$ are their respective generalized momenta (Q is a bounded connected region of the plane $\mathbf{R}^2\{I_1, I_2\}$), and μ is a small parameter. It is assumed that F is a 2π periodic function of the generalized coordinates, and analytic with respect to all its arguments in the direct product $Q \times \mathbf{T}^2 \times [0, \varepsilon)$.

The equations of motion with the Hamilton function (1.1) when $\mu = 0$ (the unperturbed system) is integrable

$$I = I^\circ,\ \varphi = \omega(I)t + \varphi^\circ \tag{1.2}$$
$$\omega = (\omega_1, \omega_2),\quad \omega_k = \partial F_0/\partial I_k\ (k = 1, 2)$$

Let the frequencies ω_1 and ω_2 of the unperturbed system be commensurable when $I = I^\circ$: $\omega_2/\omega_1 = l/m$ $(m \in \mathbf{N},\ l \in \mathbf{Z})$. Then the generating solution (1.2) is periodic with some period τ. We select the initial instant of time so that $\varphi_1 = 0$ for any μ and when $t = 0$. Suppose that the following Poincaré conditions

$$\det\left\| \frac{\partial^2 F_0}{\partial I_j \partial I_k} \right\|_{j, k=1, 2} \neq 0 \text{ for } I = I^\circ \tag{1.3}$$

$$\exists\ \varphi_2^\circ = \lambda,\ \ \partial\langle F_1\rangle/\partial\lambda = 0,\ \ \partial^2\langle F_1\rangle/\partial\lambda^2 \neq 0 \tag{1.4}$$

$$\left(\langle F_1\rangle = \frac{1}{\tau}\int_0^\tau F_1(I^\circ, \omega_1 t, \omega_2 t + \lambda)\, dt\right) \tag{1.5}$$

for the existence of periodic solutions of the perturbed system with the Hamilton function are satisfied. Then for reasonably small $\mu \neq 0$ there exists a periodic solution of period τ for the periodic system that analytically depends on parameter μ, when $\mu = 0$, becomes the periodic solution (1.2) of the unperturbed system.

We write that solution in the form

$$\varphi_1 = w_1 + \sum_{k=1}^{\infty} \mu^k \varphi_1^{(k)}(w_1) \tag{1.6}$$

$$\varphi_2 = \frac{\omega_2}{\omega_1} w_1 + \lambda + \sum_{k=1}^{\infty} \mu^k \varphi_2^{(k)}(w_1)$$

$$I_j = I_j^\circ + \sum_{k=1}^{\infty} \mu^k I_j^{(k)}(w_1),\quad j = 1, 2$$

where all functions on the right sides are periodic of period $2\pi m$ relative to the variable $w_1 = \omega_1 t$. Everywhere below the derivatives with respect to the variables I are calculated for $I = I^\circ$.

Theorem 1. Let the Hamilton function of the autonomous system with two degrees of freedom have the form (1.1), and suppose the initial conditions of the generating periodic solution (1.2) are selected so that conditions (1.3) and (1.4) of the Poincaré theorem are satisfied. If these initial values satisfy the conditions

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$$\frac{\partial^2 \langle F_1\rangle}{\partial\lambda^2}\left(\omega_1^2\frac{\partial^2F_0}{\partial I_2^2}-2\omega_1\omega_2\frac{\partial^2F_0}{\partial I_1\partial I_2}+\omega_2^2\frac{\partial^2F_0}{\partial I_1^2}\right)>0 \tag{1.7}$$

$$\partial^4\langle F_1\rangle/\partial\lambda^4\not\equiv 0 \tag{1.8}$$

the orbital stability of the periodic solution (1.6) of the perturbed system exsits.

Proof. We change to the new canonical variables w_1, q_2, r_1, p_2 $(w_1\equiv\omega_1 t)$, such that when $q_2=p_2=r_1\equiv 0$ we obtain the periodic solution (1.6). The variables q_2, p_2, r_1 are perturbations of the periodic solution (1.6). Perturbations q_2, p_2 are of the first order of smallness, and r_1, as the action variable, is a quantity of the second order of smallness.

The perturbations are defined by the formulas

$$\begin{aligned}
&\varphi_1=w_1+\sum_{k=1}^{\infty}\mu^k\varphi_1^{(k)}(w_1)\\
&\varphi_2=\frac{\omega_2}{\omega_1}w_1+\lambda+\sum_{k=1}^{\infty}\mu^k\varphi_2^{(k)}(w_1)+q_2\\
&I_1=I_1^\circ+\sum_{k=1}^{\infty}\mu^kI_1^{(k)}(w_1)+r_1-\frac{\omega_2}{\omega_1}p_2+\sum_{k=1}^{\infty}\mu^kG_*^{(k)}(w_1,q_2,r_1,p_2)\\
&I_2=I_2^\circ+\sum_{k=1}^{\infty}\mu^kI_2^{(k)}(w_1)+p_2
\end{aligned} \tag{1.9}$$

where the functions $G_*^{(k)}$ are selected so that transformation (1.9) is canonical and $G_*^{(k)}(w_1, 0, 0, 0)\equiv 0$ for any $k=1,2$, etc. For the generating function

$$S=\sum_{k=0}^{\infty}\mu^kS_k(\varphi_1,\varphi_2,r_1,p_2) \tag{1.10}$$

we have the equations

$$\frac{\partial S}{\partial r_1}=w_1,\quad \frac{\partial S}{\partial p_2}=q_2,\quad \frac{\partial S}{\partial \varphi_1}=I_1,\quad \frac{\partial S}{\partial \varphi_2}=I_2 \tag{1.11}$$

From (1.9)–(1.11) we have

$$\begin{aligned}
&S_0=\varphi_1\left(I_1^\circ+r_1-\frac{\omega_2}{\omega_1}p_2\right)+\varphi_2(I_2^\circ+p_2)-\lambda p_2\\
&S_1=-\left(r_1-\frac{\omega_2}{\omega_1}p_2\right)\varphi_1^{(1)}(\varphi_1)-p_2\varphi_2^{(1)}(\varphi_1)+\varphi_2I_2^{(1)}(\varphi_1)+\\
&\quad\int_0^{\varphi_1}\left\{I_1^{(1)}(\xi)-\left(\frac{\omega_2}{\omega_1}\xi+\lambda\right)\frac{dI_2^{(1)}(\xi)}{d\xi}\right\}d\xi\\
&G_*^{(1)}=-\left(r_1-\frac{\omega_2}{\omega_1}p_2\right)\frac{d\varphi_1^{(1)}(w_1)}{dw_1}-p_2\frac{d\varphi_2^{(1)}(w_1)}{dw_1}+q_2\frac{dI_2^{(1)}(w_1)}{dw_1}
\end{aligned}$$

To determine the stability conditions of the perturbed solution (1.6) we use Barrar's theorem /5/. The Hamilton function of perturbed motion expanded in powers of r_1, p_2, q_2, μ in the neighbourhood of initial values that generate solution (1.6) has the form

$$\begin{aligned}
&F^*=\omega_1r_1+\frac{1}{2}p_2^2D^2F_0+\frac{1}{6}p_2^3D^3F_0+\frac{1}{24}p_2^4D^4F_0+\\
&\frac{1}{2}r_1^2\frac{\partial^2F_0}{\partial I_1^2}+r_1p_2D\frac{\partial F_0}{\partial I_1}+\frac{1}{2}r_1p_2^2D^2\frac{\partial F_0}{\partial I_1}+\\
&\mu\left\{\frac{1}{2}D_1^2F_1+\frac{1}{6}D_1^3F_1+r_1D_1\frac{\partial F_1}{\partial I_1}+\frac{1}{24}D_1^4F_1+\right.\\
&\frac{1}{2}r_1D_1^2\frac{\partial F_1}{\partial I_1}+\frac{1}{2}r_1^2\frac{\partial^2F_1}{\partial I_1^2}+I_1^{(1)}(w_1)D_2\frac{\partial F_0}{\partial I_1}+\\
&\left.I_2^{(1)}(w_1)D_2\frac{\partial F_0}{\partial I_2}+G_*^{(1)}\left(p_2D+r_1\frac{\partial}{\partial I_1}+D_2\right)\frac{\partial F_0}{\partial I_1}\right\}+\ldots\\
&D=\frac{\partial}{\partial I_2}-\frac{\omega_2}{\omega_1}\frac{\partial}{\partial I_1},\quad D_1=q_2\frac{\partial}{\partial\lambda}+p_2D\\
&D_2=\frac{1}{2}p_2^2D^2+\frac{1}{6}p_2^3D^3+r_1p_2D\frac{\partial}{\partial I_1}+\\
&\frac{1}{2}r_1^2\frac{\partial^2}{\partial I_1^2}+\frac{1}{2}r_1p_2^2D^2\frac{\partial}{\partial I_1}\\
&F_0=F_0(I^\circ),\quad F_1=F_1\left(I^\circ,w_1,\frac{\omega_2}{\omega_1}w_1+\lambda\right)
\end{aligned} \tag{1.12}$$

where the dots denote terms of the order of smallness relative to perturbations higher than the fourth and, also, terms of the order of smallness relative to μ higher than the second.

The Hamiltonian of perturbed motion is a $2\pi m$ periodic function of the variable w_1.

We will represent it in the form

$$F^* = \Phi_1 + \mu\Phi_2 \tag{1.13}$$

$$\Phi_1 = \frac{1}{2\pi m}\int_0^{2\pi m} F^*(w_1, q_2, r_1, p_2)\,dw_1, \quad \langle\Phi_2\rangle \equiv 0$$

Consider the Hamiltonian of K which is the quadratic part of Φ_1 in variables q_2, p_2

$$K = \mu b q_2^2 + \mu c q_2 p_2 + (a_0 + \mu a)p_2^2 \tag{1.14}$$

$$b = \frac{1}{2}\frac{\partial^2\langle F_1\rangle}{\partial\lambda^2}, \quad c = \frac{\partial^2\langle F_1\rangle}{\partial\lambda\partial I_2} - \frac{\omega_2}{\omega_1}\frac{\partial^2\langle F_1\rangle}{\partial\lambda\,\partial I_1}$$

$$a_0 = \frac{1}{2}D^2F_0 \equiv \frac{1}{2\omega_1^2}\left(\omega_1^2\frac{\partial^2 F_0}{\partial I_2^2} - 2\omega_1\omega_2\frac{\partial^2 F_0}{\partial I_1\partial I_2} + \omega_2^2\frac{\partial^2 F_0}{\partial I_1^2}\right)$$

$$a = \frac{1}{2}D^2\langle F_1\rangle + \frac{1}{2}\langle I_1^{(1)}\rangle D^2\frac{\partial F_0}{\partial I_1} + \frac{1}{2}\langle I_2^{(1)}\rangle D^2\frac{\partial F_0}{\partial I_2}$$

$$\left\|\begin{matrix}\langle I_1^{(1)}\rangle\\ \langle I_2^{(1)}\rangle\end{matrix}\right\| = -\left\|\frac{\partial^2 F_0}{\partial I_k\partial I_j}\right\|^{-1}\left\|\frac{\partial\langle F_1\rangle}{\partial I_k}\right\| \quad (k, j = 1, 2)$$

Below, we assume that $a_0 \neq 0$, which means that the level lines of the function $F_0(I)$ have no inflections in the region Q. The characteristic equation corresponding to (1.14) has the form

$$\alpha^2 + 4\mu b\,(a_0 + \mu a) - \mu^2c^2 = 0$$

Let us assume that $4a_0b > 0$. Then the characteristic equation has two purely imaginary complex-conjugate roots

$$\alpha_{1,2} = \pm\, i\sqrt{\mu}\Omega_2 \equiv \pm\, i\,[4\mu b\,(a_0 + \mu a) - \mu^2c^2]^{1/2}$$

Otherwise the periodic solution (1.6) is unstable.

As the result of a canonical transformation

$$w_1 = w_1', \quad q_2 = \mu^{1/4}\beta_1\sqrt{2r_2'}\sin w_2' - \mu^{1/4}\beta_2\sqrt{2r_2'}\cos w_2'$$

$$r_1 = \mu r_1', \quad p_2 = \mu^{1/4}\beta_1^{-1}\sqrt{2r_2'}\cos w_2'$$

$$(\beta_1 = \operatorname{sign} b\sqrt{\Omega_2/2\,|b|}, \quad \beta_2 = c\,(2\,|b|\,\Omega_2)^{-1/2})$$

of valency $1/\mu$, we obtain a new Hamiltonian of the perturbed motion, which we represent in the form

$$F^{**} = \omega_1 r_1 + \sqrt{\mu}\,\{K_1(r_1, r_2) + K_2(w_1, w_2, r_1, r_2)\} \tag{1.15}$$

$$K_1(r_1, r_2) = \frac{1}{2\pi m}\int_0^{2\pi m}\Phi_1(w_2, r_1, r_2)\,dw_2$$

$$\frac{1}{(2\pi m)^2}\int_0^{2\pi m}\int_0^{2\pi m} K_2(w_1, w_2, r_1, r_2)\,dw_1\,dw_2 \equiv 0$$

(primes on the new variables are omitted). Transformation (1.15) enables us to consider the change of variable r_2 in the ring $V_2 = \{\rho_1 \leqslant r_2 \leqslant \rho_2,\ \rho_1, \rho_2 > 0\}$. The Hamiltonian F^{**} is then an analytic function of all its arguments in the direct product $V \times \mathbf{T}^2 \times [0, \varepsilon)$, where $V = V_1 \times V_2$, $V_1 \subset \mathbf{R}^1\{r_1\}$ and V_1, V_2 are closed sets.

We have

$$\sqrt{\mu}\,K_1(r_1, r_2) = \sqrt{\mu}\operatorname{sign} b\Omega_2 r_2 + \mu A_1 r_1^2 + \mu A_2 r_2^2 + \mu^{3/2}A_{12}r_1r_2 + O(\mu^2, r_kr_j^2) \quad (k, j = 1, 2)$$

$$A_1 = \frac{1}{2}\frac{\partial^2 F_0}{\partial I_1^2}, \quad A_2 = \frac{1}{16}\beta_1^4\frac{\partial^4\langle F_1\rangle}{\partial\lambda^4}$$

$$A_{12} = \frac{1}{2}\left(D^2\frac{\partial F_0}{\partial I_1} + \beta_1^2\frac{\partial^3\langle F_1\rangle}{\partial I_1\partial\lambda^2}\right)$$

Consider the determinant

$$\det\left\|\begin{matrix}\dfrac{\partial^2(\omega_1r_1 + \sqrt{\mu}\,K_1)}{\partial r_k\partial r_j} & \dfrac{\partial(\omega_1r_1 + \sqrt{\mu}\,K_1)}{\partial r_k}\\ \dfrac{\partial(\omega_1r_1 + \sqrt{\mu}\,K_1)}{\partial r_j} & 0\end{matrix}\right\| = \mu N = -\mu\omega_1^2A_2 + O(\mu^2, r_k) = -\mu\omega_1^2\frac{1}{16}\beta_1^4\frac{\partial^4\langle F_1\rangle}{\partial\lambda^4} + O(\mu^2, r_k), \quad k, j = 1, 2$$

If $N \not\equiv 0$, the Hamiltonian F^{**} of perturbed motion satisfies all conditions of Barrar's theorem, and hence the periodic solution (1.6) is orbitally stable. The conditions of orbital stability of solution (1.6) thus have the form (1.7).

2. Let us use the above theorem to investiage the orbital stability of Poincaré periodic solutions in the problem of the motion of a heavy solid about a fixed point, which were obtained in /6, 7/. Following /8/, we shall show that in this problem the Poincaré periodic solutions that are stable in the linear approximation /7/, are orbitally stable.

The Hamilton function in the problem of a heavy solid rotating about a fixed point, in Andoyer canonical variables $L, G, H, \quad l, g, h$, has the form

$$F = F_0 + \mu F_1, \qquad F_0 = \frac{(G^2 - L^2)}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right) + \frac{L^2}{2C}, \quad F_1 = \frac{x}{r}\gamma_1 + \frac{y}{r}\gamma_2 + \frac{z}{r}\gamma_3 \tag{2.1}$$

$$\gamma_1 = \Gamma_1 \sin l + \Gamma_2 \sin l \cos g + (G/L)\,\Gamma_2 \cos l \sin g, \quad \gamma_2 = \Gamma_1 \cos l + \Gamma_2 \cos l \cos g - (G/L)\,\Gamma_2 \sin l \sin g$$

$$\gamma_3 = \Gamma_1\Gamma_2 \cos g; \quad \Gamma_1 = \frac{H}{G}\sqrt{1 - \left(\frac{L}{G}\right)^2}, \quad \Gamma_2 = \frac{L}{G}\sqrt{1 - \left(\frac{H}{G}\right)^2}$$

where F_0 is the Hamiltonian in the Euler-Poinsot case. The regions of possible values of L and G is the set $\Delta = \{(L, G): G \geqslant 0, \ |L| \leqslant G\}$, A, B, C are the principal moments of inertia of the body with $(A \geqslant B \geqslant C)$; μ is the small parameter equal to the body weight multiplied by the distance between the centre of mass and the suspension point, (x, y, z) are the coordinates of the centre of mass in the principal axes of the body ellipsoid of inertia, and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the centre of mass and the suspension point.

Since the Hamiltonian F is independent of the variable h, the momentum H corresponding to that variable is the integral of motion (the area integral). By fixing the constant of the area integral $H = H_0$, we reduce the problem considered here to a system with two degrees of freedom.

In the case of dynamic symmetry $A = B$ one can assume that $y = 0$, then the Hamiltonian function (2.1) takes the form

$$F = \frac{1}{2A}G^2 + \frac{1}{2}\left(\frac{1}{C} - \frac{1}{A}\right)L^2 + \mu\left\{\frac{x}{r}\gamma_1 + \frac{z}{r}\gamma_3\right\} \tag{2.2}$$

When $\mu = 0$, we have the following generating motion:

$$G = G_0, \quad L = L_0, \quad g = \omega_1 t, \quad l = \omega_2 t + l_0, \quad \omega_1 = \frac{G_0}{A}, \quad \omega_2 = \left(\frac{1}{C} - \frac{1}{A}\right)L_0$$

From the results of /7/ and Theorem 1 on the stability of periodic solutions we have the following theorems (cf. /8/).

Theorem 2. Let $x \neq 0$ and $A = B > 2C$. Then, on two-dimensional invariant tori

$$\frac{G}{A} = \pm\left(\frac{1}{C} - \frac{1}{A}\right)L, \quad G \neq 0, \quad G \neq |H_0|$$

of the Euler-Poinsot problem pairs of isolated periodic solutions of the perturbed system are generated for small $\mu \neq 0$. These solutions analytically depend on μ, and one of each pair of solutions is orbitally stable, and the other unstable.

Theorem 3. Let $x \neq 0$, $A = B \neq C$ and $H_0 \neq 0$, $G \neq |H_0|$. Then on resonance tori $G = G_0 > 0$, $L = 0$ (the rotations are around the principal axes of inertia in the equatorial plane of the ellipsoid of inertia) of the Euler-Poinsot problem pairs of isolated periodic solutions of the perturbed system are generated for small values of parameter $\mu \neq 0$. They analytically depend on μ, and one solution of each pair is orbitally stable, and the other unstable.

In the case of an unsymmetric solid $A > B > C$ we introduce in the Euler-Poinsot problem the "action" variables

$$I_3 = I_3^\circ = H_0, \quad I_2 = G, \quad I_1(I_2, F_0) = \frac{1}{2\pi}\int_0^{2\pi}\left[2F_0 - I_2^2\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)\right]^{1/2}\left[\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right]^{-1/2} dl \tag{2.3}$$

The variables φ_1, φ_2 conjugate to I_1, I_2 are expressed in terms of l, g by elliptic quadratures. In I_1, I_2 coordinates again the region $\Delta = \{(I_1, I_2): I_2 \geqslant 0, \ |I_1| \leqslant I_2\}$. Expansion of the perturbing function $F_1(I_1, I_2, I_3^0, \varphi_1, \varphi_2)$ in a double Fourier series in the variables φ_1, φ_2 has the form

$$F_1 = \sum_{-\infty}^{\infty} F_{m,1}\exp[i(m\varphi_1 + \varphi_2)] + \sum_{-\infty}^{\infty} F_{m,-1}\exp[i(m\varphi_1 - \varphi_2)] + \sum_{-\infty}^{\infty} F_{m,0}\exp[im\varphi_1] \tag{2.4}$$

We will introduce into the analysis the sets

$$\Delta_a = \Delta \setminus (\{I_1 = 0\} \cup \{2F_0 = I_2^2/B\} \cup \{|I_1| = I_2\})$$
$$\Delta^\circ = \Delta \cap \{(I_1, I_2): |I_3^\circ| < I_2\}$$
$$\Delta_a^\circ = \Delta_a \cap \{(I_1, I_2): |I_3^\circ| < I_2\}$$
$$V = \{I = (I_1, I_2): I \subset \Delta^\circ;\ m\omega_1(I) \pm \omega_2(I) = 0,$$
$$\omega_j = \partial F_0/\partial I_j\ (j = 1, 2);\ m \in Z \setminus \{0\};\ F_{m,\pm 1}(I) \neq 0\}$$

where V is the secular set of the perturbed system. It was shown in /8/ that the function $F_0(I)$ is continuous in the region Δ, and is homogeneous of power 2; it is analytic in the region Δ_a, non-degenerate (the Hessian of $F_0(I)$ in variables I_1, I_2 is non-zero), and isoenergetically non-degenerate (the level lines of $F_0(I)$ have no inflections in Δ_a), and the function $F = F_0 + \mu F_1$ is determinate and analytic in Δ°_a. It was shown in /8/ that expansion (2.4) has an infinite number of coefficients of the form $F_{m,\pm 1}$ that are non-zero for $I \in V$.

Theorem 4. Let $I = I^\circ \in V$, $V \subset \Delta^\circ$ be the secular set of the perturbed system. Then from the set of periodic solutions of the Euler-Poinsot problem that lie on the torus $I = I^\circ \in \Delta^\circ_a$ at least tow isolated periodic solutions are generated when there is a perturbation. These solution exist for fairly small $\mu \neq 0$ and depend analytically on μ. One of the solutions is then orbitally stable, and the other unstable.

A proof of existence of the Poincaré periodic solutions was given in /8/. We have to show that solutions that are stable in the linear approximation, are orbitally stable. As an example, let us consider Theorem 4. We set

$$\varphi_1 = \omega_1 t,\ \varphi_2 = \omega_2 t + \lambda,\ \omega_j = \partial F_0/\partial I_j\ (j = 1,2) \tag{2.5}$$
$$I = I^\circ = (I_1, I_2) \in \Delta_a^\circ,\ I^\circ \in V,\ \omega_2/\omega_1 = m$$

From (2.4) it follows that

$$\langle F_1 \rangle = F_{-m,1} e^{i\lambda} + F_{m,-1} e^{-i\lambda} + F_{0,0} \tag{2.6}$$

and for some λ we have $\partial \langle F_1 \rangle / \partial \lambda = 0$ /8/ and

$$\frac{\partial^2 \langle F_1 \rangle}{\partial \lambda^2} = -F_{-m,1} e^{i\lambda} - F_{m,-1} e^{-i\lambda} \neq 0$$

It is evident from (2.6) that $\partial^4 \langle F_1 \rangle / \partial \lambda^4 = -[\partial^2 \langle F_1 \rangle / \partial \lambda^2 \neq 0$, consequently, the Poincaré periodic solutions that are stable in linear approximation, are orbitally stable.

Theorems 2 and 3 can be proved similarly.

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